Stability properties of discrete stock-production models*

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ABSTRACT: The paper focuses on the stability analysis of a certain class of catch- and effort-controlled discrete stock-production models for optimal management of exploited populations, which were used as official methods for Cape hake assessments in the International Commission for the Southeast Atlantic Fishery (ICSEAF) during 1988-1990, and are still used for forecasting TACs of this species and other commercially fished stocks. In spite of the formal approach used, the problem is of the utmost practical importance as instability of a model raises the inaccuracy of the resultant estimates, providing also failures at the stage of the model fitting and total allowable catch (TAC) forecasting. So, clarification of the stability conditions of stock-production models enables their more deliberate and effective use in management of marine living resources. The effort-controlled models exhibit higher stability as compared with the catch-controlled ones: the fishing effort acts as a recovering force which suppresses the perturbations initiated in the model by the inevitable errors in estimating the initial population level. Therefore, the optimal management strategies and the TAC forecasts derived from the effort-controlled models are less risky than those obtained with the use of the catch-controlled ones.

Key words: Stock-production models, assessment of fisheries, fisheries management

RESUMEN:ESTABILIDAD DE LOS MODELOS DE PRODUCCIÓN discretos. – En este trabajo se realiza un análisis de la estabilidad de un tipo de modelos de producción, con control de captura o esfuerzo, con el fin de optimizar la gestión óptima de las poblaciones explotadas. Tales modelos han sido empleados como métodos oficiales para las evaluaciones de la merluza del Cabo en la International Commission for the Southeast Atlantic Fishery (ICSEAF) entre 1988 y 1990, y son todavía usados en el cálculo de los TACs de esta especie y de otros recursos explotados. Independientemente del método formal empleado, el problema de la estabilidad es de gran importancia, ya que la inestabilidad del modelo aumenta la imprecisión de los resultados, causando fracasos a nivel de estimación de la captura máxima (TAC). Así la clarificación de las condiciones de estabilidad de los modelos de producción facilita su mejor y más efectivo uso en la gestión de los recursos marinos vivos. Los modelos de control del esfuerzo muestran una mayor estabilidad que los basados en el control de la captura: el esfuerzo de pesca actúa como una fuerza de recuperación que elimina las perturbaciones iniciadas en el modelo por los inevitables errores de estimación del nivel inicial de población. Por lo tanto, las estrategias óptimas de gestión y de prevención de TAC derivados de los modelos de control de esfuerzo tienen menos riesgo que aquellos que se controlan mediante la captura (Traducido por el Editor).

Palabras clave: Modelos de producción, evaluación de pesquerías, gestión de pesquerías.

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INTRODUCTION

The problem of assessment of an exploited stock size and evaluation of quota or total available catch (TAC) is the central one in management of commercial fishery. The aim of such a control is to optimize the allowed yield under the strong condition of keeping the natural living resource in the state of nearly highest reproduction during sufficiently long period, and of reducing the risk of its overexploitation. Mathematical models of dynamics of exploited populations (along with biological analysis of the population state and dynamics) serve as a theoretical basis for the optimal management. When dealing with this problem one has to sometimes operate with commercial fishery data which do not reflect the age structure of the exploited population. In such a case a stock-production model can serve as a mathematical instrument of the investigation. While only traditional (i.e. equilibrium) models were in use, it was possible to discuss the reliability and accuracy of the estimates obtained, but the question of their stability did not arise. But in the past two decades, that the dynamic stock-production models were being introduced to TAC forecasting, the question became not only meaningful but a very important one, constituting (for such models) a significant supplement to risk analysis, which has been defined as "...the evolution of the probability of end events interpreted in terms of sequences of earlier events" (Linder et al., 1987).

A dynamic stock-production model describes changes with time of the population size (biomass or number of individuals) or of its index, such as catch per unit effort (CPUE), under the controlling action of fishery. There are two kinds of dynamic models. The basis of a continuous model is constituted by a first order differential equation, while a discrete model, describing year-to-year dynamics, is based on a recurrent formula, which may be regarded as a finite difference analog of the basic differential equation of the continuous model. Naturally, in order to uniquely determine a continuous or discrete solution of the corresponding equation, an initial condition (a starting value) for the population size or for CPUE must be set. Any perturbation in the starting value, i.e., a deviation from the initial condition determining a solution, will cause a difference between the new and the original solutions at every next moment. In other words, perturbation in the starting value will perturb the whole solution. The original, unperturbed solution is called stable if the perturbation remains limited within a sufficiently long (strictly speaking, infinitely long) time period, otherwise it is called unstable. The perturbation may happen to be not only limited, but decaying with time, in which case the solution is called asymptotically stable. All stable solutions, which are not asymptotically stable, are called neutrally stable. Correspondingly, one may speak of unstable and stable (neutrally and asymptotically stable) models.

Instability of a model raises the inaccuracy of the resultant estimates, providing also failures at the stage of the model fitting and TAC forecasting. So, clarification of the stability conditions of stock-production models, which is the subject of the paper, enables their more deliberate and effective use. Stability of stationary states of continuous effort-controlled production models with different stock-production functions had been examined by May et al. (1978) in the terms of ‘characteristic return time’, the reciprocal rate at which small perturbations die away (see also Schaefer, 1954; Beddington and May, 1977).

The present paper is focused on the stability analysis of the discrete catch- and effort-controlled models suggested by the author (Kizner, 1989, 1990; see also Kizner, 1992). However, in the same way the stability of the catch-controlled model by Butterworth and Andrew (1984) along with its effort-controlled modification (which can be constructed after the analogy of the effort-controlled model by Kizner) can be analyzed providing exactly the same results. These models were used as official methods for Cape hake TAC assessment in the International Commission for the Southeast Atlantic Fishery (ICSEAF) in 1988 - 1990. Some of them are still in use for forecasting TACs of Cape hake and other commercially fished stocks (Vasilyev et al. 1990; Punt, 1994).

STABILITY ANALYSIS

Catch-controlled model

Two equations expressing balance of the stock biomass and the proportion between the biomass and CPUE:

\[ B_{t+1} = B_t + P(B_t) - C_t, \]

\[ v_t = qB_t, \]

will serve as a basis for the following constructions (Kizner, 1989, 1990). Here
\( B_t \) is biomass at start of the year \( t \);
\( v_t \) is CPUE at start of the year \( t \);
\( C_t \) is catch in the year \( t \);
\( P \) is production function: \( P(B_t) = rB_t (1-B_t/K) \) and \( P(B_t) = rB_t (1-lnB_t/lnK) \) according to Schaefer* (1954) and Fox (1970) respectively;
\( q, r, K \) are positive constants:
- \( q \) catchability coefficient,
- \( r \) intrinsic growth rate,
- \( K \) carrying capacity of the environment.

Here and below in this section we operate only with ‘model’ (estimated) variables, except for catch, \( C_t \), thus there is no need to mark them with any special sign. The ‘estimated catch’, \( \hat{C}_t \), will be introduced in the next section.

Substitution of (2) into (1) reduces the system (1), (2) to one equation with respect to CPUE:

\[
v_{t+1} = v_t + qP \left( \frac{v_t}{q} \right) - qC_t . \tag{3}
\]

This model may be called the model with the control through catch, \( C_t \), which is supposed to be known for a period \( t = 1,...,n \). The ‘model’ (estimated) CPUE dynamics during the period of fishing history can be evaluated by means of a chain of recurrent calculations through the equation (3) under the condition that its starting value, \( v_1 \), is known.

First, stability of the model with the Schaefer production function will be examined. In this case the equation (3) takes the form:

\[
v_{t+1} = v_t + r v_t \left( 1 - \frac{v_t}{qK} \right) - qC_t . \tag{4}
\]

Let us choose a solution \( V_t \) of equation (4) and study the conditions of its stability. Suppose the starting value, which determines the solution \( V_t \), is perturbed. The new solution, \( v_t \), corresponding to the new initial condition, will be called the perturbed one, while the difference, \( \varepsilon_t = v_t - V_t \), will be referred as the perturbation of \( V_t \). Substituting the expansion

\[
v_t = V_t + \varepsilon_t , \tag{5}
\]

for \( v_t \) into (4) and omitting the quadratic term, \( r^2 v_t^2 / qK \), one gets the following simple relationship between \( \varepsilon_{t+1} \) and \( \varepsilon_t \):

\[
\varepsilon_{t+1} = p_t \varepsilon_t , \tag{6}
\]

where

\[
p_t = \left[ 1 + r \left( 1 - \frac{2V_t}{qK} \right) \right] \tag{7}
\]

For details see Appendix.

Linearization, i.e. omitting the terms of the order higher than one, is always legitimate if the perturbation is small. Its validity under general conditions (that is for non-small perturbations) can be substantiated by the following reasoning. Equation (3) may be regarded as a difference approximation of the differential equation

\[
\frac{dv}{dt} = qP \left( \frac{v}{q} \right) - qC(t) , \tag{8}
\]

where \( C(t) \) is the catch rate. When \( P \) is the Schaefer or Fox function, linear stability analysis is allowed for the study of the behavior of non-small perturbations of the solutions of equation (8) (see Appendix). Therefore it is quite natural to assume that linearization is also applicable for checking the growth or decrease of non-small perturbations when the stability of the discrete model (3) is studied. Computer experiments with the models under consideration support this assumption.

Obviously, in accordance with (6), the perturbation \( \varepsilon_t \) does not grow while \( |p_t| \leq 1 \). The intrinsic growth rate, \( r \), is determined by both the input data used for the model fitting, and the accepted time scale, \( T^* \). Usually, \( r < 1 \) if \( T^* = 1 \) yr. (e.g., for the Cape hakes \( r \) varies from about 0.3 to 0.6 depending on the input data and the type of the model). On the other hand, only in the early years of the period of intensive commercial fishery the CPUE may turn out to be close to the saturation level, \( qK \), while in the other years \( V_t < gK \). Therefore, in the context of the stability analysis of the model one undoubtedly may assume that \( r(1-2V_t/qK) > -1 \) in (7). It means that \( p_t \) is positive, and the perturbation keeps its sign.

It is clear from (7) that there exists a critical level of the unperturbed CPUE, namely, \( V_{CR} = qK/2 \), that delimits the stability and instability regions, the crit-
The critical level itself being the level of neutral stability. Indeed, $1 - 2V_t /qK \leq 0$ and consequently $0 < p_t \leq 1$ when $V_t \geq V_{CR}$. On the contrary, $1 - 2V_t /qK > 0$ and $p_t > 1$ when $V_t < V_{CR}$. In terms of biomass, $V_{CR}$ corresponds to $B_{MSY} = K/2$, the biomass at maximum surplus production level that provides maximum sustainable yield (MSY). Thus, we may conclude that the perturbation does not grow while $V_t \geq V_{CR}$, whereas it does grow monotonously while $V_t < V_{CR}$. In the both cases it is supposed that $V_t$ does not approach $V_{CR}$ asymptotically. If these conditions are maintained, the perturbation grows or decreases no slower than a geometric progression. Practically, all these mean that when within a sufficiently long time period the graph of the estimated CPUE versus time passes above the critical level, $V_{CR}$, the solution, $V_t$, corresponding to this period of relatively weak fishing is asymptotically stable, while below $V_{CR}$, that is, when the population is strongly fished, the instability may distort the true dynamics.

In order to illustrate such kind of instability, the following computer experiment had been performed. First, the model was fitted using the data on Cape hake fishery during 1966 - 1989 for the ICSEAF Division 1.5, and so, its parameters $K \approx 1.2 \times 10^2 (10^3 t)$, $r \approx 0.56$ and $q \approx 1.7 \times 10^{-3} (10^3 h^{-1})$ were determined providing the best description of the CPUE data by the model according to the Least Squares Principle (for the fitting procedure see Kizner, 1992). At this stage, the chain of calculations through equation (3) started from 1967, the average actual CPUE in 1996 and 1967 being used as a starting value. The resultant series of estimated CPUEs turned out to fit the real data very closely, and we refer it as an ‘unperturbed’ solution, $V_t$, in the experiments with the catch-controlled model (Figure 1). Then the start point for the recurrent calculations of the $V_t$ series was moved to 1973, while the former values of the parameters $r, K,$ and $q$ were kept. If we were starting the new run from $V_{1973}$ then the obtained series, $V_{1973},..., V_{1989}$ were exactly equal to the best fit estimates, $V_{1973},..., V_{1989}$. We, however, have used the actual CPUE in 1973 as a starting value for calculating $V_{1973},..., V_{1989}$. Since the new starting CPUE differed a bit from the ‘unperturbed’ value, $V_{1973}$, such a choice provided a small initial perturbation in the model. The subsequent monotonous growth of the perturbation can be observed in Figure 1.

Because of the generality of the conclusion regarding the regions of stability and instability, it remains valid for stationary states of the model under consideration. So, we can state that every sustainable yield state, i.e. equilibrium between the population surplus production and yield, is asymptotically stable, neutrally stable, or unstable, if the population biomass is higher, equal, or less than $B_{MSY}$ respectively.

As it was noted above, in the case of $P$ being the Fox function, the linear approach is also applicable for stability analysis of the model (3) without restricting ourselves by considering only small perturbations (see Appendix). This requires the use of the power expansion of the logarithmic function and leads to qualitatively the same results. In particular, the critical level is the level of maximum surplus production $V_{MSY} = qK/e$ (or $B_{MSY} = B_{MSY} = K/e$) providing MSY. Moreover, it can be shown that in the case of an arbitrary production function $P$ with one maximum, the critical level for the stability in the terms of biomass is $B_{MSY}$ (but, strictly speaking, such a general statement is valid for small perturbations only).

Essentially the same results by a similar reasoning can be obtained regarding the stability of the Butterworth and Andrew (1984) model.

The procedure of TAC forecasting and its stability are examined below.

**Effort-controlled model**

Another model (Kizner, 1989, 1990), the model with the control through fishing effort, derives from
the previous one after replacing in (3), (4) and (8) the real catch, \( C_t \), by the estimated one, \( \hat{C}_t \). As the real catch is a product of the real CPUE and effort, the estimated catch should be constructed in a similar manner, but using the estimated CPUE. According to its definition, the estimated CPUE, \( v_t \), corresponds to the start of the year \( t \), while \( f_t \) and \( \hat{C}_t \) represent respectively the fishing effort and estimated catch during the entire year \( t \). Therefore, when defining \( \hat{C}_t \) it is reasonable to make it dependent on both \( v_t \) and \( v_{t+1} \):

\[
\hat{C}_t = \frac{v_t + v_{t+1}}{2} \cdot f_t. \tag{9}
\]

Now it is the fishing effort rather than the catch that becomes the external control action upon the stock. Substituting (9) for the true catch in dynamics equation (3) changes its type considerably. When \( P \) is the Schaefer function, the governing equation is:

\[
v_{t+1} = v_t + rv_t \left( 1 - \frac{v_t}{qK} \right) \cdot \frac{qf_t}{2} \cdot \frac{v_t + v_{t+1}}{2}, \tag{10}
\]

which gives:

\[
v_{t+1} = \frac{1 - qf_t}{2 + r (1-v_t/qK)} \cdot \frac{v_t + qf_t}{2}, \tag{11}
\]

Acting in accordance with the above described scheme and substituting (5) for \( v_t \) into (11), one obtains in the linear approximation the following recurrent equation for the perturbations:

\[
\varepsilon_{t+1} = \left\{ 1 + \frac{2r}{2 + qf_t} \left[ \left( 1 - \frac{qf_t}{r} \right) - \frac{2v_t}{qK} \right] \right\} \varepsilon_t. \tag{12}
\]

Again, analogy with the corresponding continuous model serves as a substantiation for the applicability of the linear approach to the study of general (not only linear) stability of the model (11), i.e. of the behavior of perturbations \( \varepsilon_t \), which are not necessarily small (see Appendix).

Strictly speaking, any critical level, that is a constant bound, does not exist in this case: it is obvious from (12) that the perturbation does not grow now when \( v_t \geq \frac{qK}{2} \cdot (1 - \frac{qf_t}{r}) \) (the upper boundary is not given because, as it was argued above, in reality \( v_t < qK \)). Therefore the notion of ‘critical bound-

\[
V_t^b, \text{ which we define as } V_t^b = \frac{qK}{2} \cdot (1 - \frac{qf_t}{r}),
\]

now replaces the notion of ‘critical level’. It is the critical boundary that delimits the asymptotic stability and instability regions of the effort-controlled model.

It is clear that the critical boundary itself and the values of \( V_t \), that satisfy the inequality \( V_t < V_t^b \), lie lower than \( qK/2 \); the higher the fishing effort the lower the critical boundary falls, and the narrower the instability zone becomes. E.g., for the Cape hake of the ICSEAF Division 5 the fitting of the model (Kizner, 1992) provides \( K \approx 1.2 \times 10^3 (10^3 t) \), \( r \approx 0.55 \), \( q \approx 1.4 \times 10^{-3} (10^{-3} h^{-1}) \), while \( qf_t \) varies approximately from 0.07 to 0.7. Hence the critical boundary lies lower than \( qK/2 \approx 0.75 \). In fact it lies lower than the actual and the estimated CPUE values for the majority of real intensively exploited stocks, and for the Cape hake among them (Figure 2). This is due to the fact that in the effort-controlled model the fishing effort acts as a recovering force which suppresses the perturbations initiated in the model by the inevitable errors in estimating the starting population level. The instability could appear in the present model only in such a hardly probable situation when the stock was diminished to a very low level as compared with the carrying capacity, and then the fishery intensity was reduced greatly within a short period. So, the model is actually always stable, which means that a certain level of errors in the initial CPUE starting the chain of recurrent calculations (11), \( v_{st} \), is admissible. It
also enables a reduction in the number of model parameters by determining $v_{st}$ directly through the first actual (observed) CPUEs instead of estimating it along with $q$, $r$, and $K$ within the fitting procedure as suggested by Butterworth and Andrew (1984).

Regarding the steady states of the model, $V = V(f)$, which are achieved when $f_t = f = \text{const}$ and $t \to \infty$, it can be easily shown that $V > V_0$ if $0 \leq f < r/q$ (Fig. 2). So, all of the steady states of the effort-controlled model are asymptotically stable except for $V = 0$ corresponding to $f = r/q$, which is neutrally stable.

The results of the stability analysis carried out are valid also when the Fox surplus production function is used.

As soon as a stock - production model is fitted, the optimal management strategy can be evaluated and presented in terms of optimal fishing effort, $f^*$, irrespective of whether the model is the catch- or effort-controlled. One of the conventional methods of adaptive fishery management is based on yearly estimation of the optimal fishing effort, $f^*$, and subsequent evaluation of the target level of the population biomass (or CPUE). The recommended TACs for a few next years are determined in such a case as a product of $f^*$, and the corresponding estimated CPUEs for these years regardless of the type of the model (see, e.g. Babayan and Kizner, 1988). Following this idea we calculate TAC forecasts as

$$\text{TAC}_{n+m} = \frac{(v_{n+m} + v_{n+m+1})}{2} f^*.$$  

If, for example, $P$ is the Schaefer function,

$$v_{n+k+1} = \frac{1-qf^*/2 + r(1-v_{n+m+1}/qK)}{1+qf^*/2} v_{n+k},$$  

(13)

for $k = 1, \ldots, m$ (the first forecasted CPUE value, $v_{n+k}$, is determined by (4) or (11)). Similarly, the prognostic equation can be derived from (3) when $P$ is the Fox function. Owing to the stability of steady states of the effort-controlled model, formally speaking, the forecasts are stable. But the TAC forecasts for the years next to $n$ experience an influence of the previous dynamics: if $v_n$ is strongly perturbed due to instability, the accuracy of the successive prognostic TACs derived with the use of (13) may turn out to be insufficient. In view of the analysis carried out, such a problem may arise in the catch-controlled model, but never in the effort-controlled one.

**CONCLUSIONS**

The TAC forecasts depend on the level of estimated CPUE for the end of the historical period, since it serves as a starting level for the forecasts, and the errors in determining this level do depend on whether the solution describing the CPUE dynamics is stable or not. Therefore, when considering the reliability of the forecasts, one has to take into account not only the stability or instability of the forecasts themselves but the stability characteristics of the dynamic model, used at the previous stage for describing the history of the fishery, as well.

The analysis carried out shows that the models with the control through fishing effort are preferable from the point of view of their stability, and therefore from the practical point of view.

The high stability of the effort-controlled model means that perturbations in the estimated CPUEs do not grow, and so a certain level of error in the starting value, $v_{st}$, is admissible: if $v_{st}$ includes a small error, the subsequent error (perturbation) will stay small for every $t$. This enables a reduction in the number of model parameters from four to three by determining $v_{st}$ directly through the first actual (observed) CPUEs instead of estimating it along with $q$, $r$, and $K$ within the fitting procedure.

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APPENDIX

Linearization

Equations (6) and (7) for the catch-controlled model and equation (12) for the effort-controlled model, as well as their analogs with the production function of Fox, can be derived following the standard procedure of linearization. Let us present the governing equations of the two models in a common form:

\[ V_{t+1} = V_t + qP\left(\frac{V_t}{q}\right) \]  \quad for CCM, \quad V_{t+1} = V_t + qP\left(\frac{V_t}{q}\right) - \left(\frac{q}{q}'\right)\left(\frac{V_t + V_{t+1}}{2}\right), \quad for ECM, \quad (A1)

where ‘CCM’ and ‘ECM’ are abbreviations of ‘catch-controlled model’ and ‘effort controlled model’ respectively. Substituting the expansion \( q = \frac{V_{t+1} - V_t}{q} \) in the right-hand part of equation (A2) and then cancel out all the linear terms, containing the factors \( V_t \) and \( V_{t+1} \), along with \( qP(V_t/q) \). It is possible to solve \( V_t \) is a solution of equation (A1). This yields the following equation for the perturbation \( \xi \):

\[ c_{t+1} = c_t + qP\left(\frac{V_t}{q}\right) - qP\left(\frac{V_t}{q}\right) - \left(\frac{q}{q}'\right)\left(\frac{V_t + V_{t+1}}{2}\right) = 0, \quad for CCM \quad (A3) \]

\[ (t\xi + \xi_{t+1})/2, \quad for ECM. \]

If we now expand the function \( P(V_t/q + \xi_t/q) / q \) in a power series with respect to \( \xi_t/q \) and omit all the terms of the order higher than one (like \( q^2K \) in the case of the Shaefer function), the linearized equation determining the series \( \xi_t \) will be obtained. For the Shaefer-type model this procedure leads to equations (6) and (7) if the control is carried out through catch, or to equation (12) in the case of the control through effort.

Linear analysis of the stability of the solutions of the differential equation (8) also includes (a) expansion of the perturbed solution \( V \) into the sum \( V = V + \xi \) of unperturbed solution \( V \) and perturbation \( \xi \) and (b) omitting the non-linear term \( \phi(\xi) \) in the corresponding equation describing the perturbation. Regardless of whether \( C(t) \) is the actual catch rate (catch-controlled model) or its theoretical estimate (effort-controlled model), in the case of the Schaefer model the linear component is \( r(1-2V/qK) \xi \) and the non-linear term is \( \phi(\xi) = r^2K \xi \). In the case of the Fox function the linear component is \( r \frac{1}{\ln K} \), while the nonlinear term may be presented as \( \phi(\xi) = -r \frac{V}{\ln K} - \omega \left(1+\alpha\right)\ln(1+\alpha) \), where \( \omega = \frac{V}{\ln K} \). A simple analysis of the behavior of the function \( \phi(\xi) \) shows that the following inequality is fulfilled for both the Schaefer- and the Fox-type models:

\[ \phi(\xi) \leq M + \alpha \xi, \quad (A4) \]

where \( M \) and \( \alpha \) are positive constants. According to the classic theory of stability of differential equations, inequality (A4) guarantees the applicability of linear stability analysis of equation (8) even when \( \xi \) is not supposed to be small, i.e. if the general (nonlinear) stability of its solutions is studied (see, for example, Bronshtein and Semendyayev, 1965; see also Coppel, 1965). Based on this, one may assume linearization to be applicable for analyzing the growth or decrease of non-small perturbations in discrete models governed by equation (A1).